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Entropy Densities for Gibbs States and Algebraic States (A joint work with D. Petz)

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1. Entropy densities for Gibbs states

First we briefly survey Gibbs states of quantum spin systems. See [4] for more details. A quantum spin system over ν-dimensional cubic lattice $\mathbf{Z}^\nu$ is described as the infinite tensor product C^*-algebra $\mathcal{A} = \bigotimes_{x\in\mathbf{Z}^\nu} \mathcal{A}_x$, $\mathcal{A}_x = M_d(\mathbf{C})$ being the $d \times d$ matrix algebra; namely $\mathcal{A}$ is the norm completion of $\bigcup_\Lambda \mathcal{A}_\Lambda$, the union of all local algebras $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A}_x$ for finite regions $\Lambda \subset \mathbf{Z}^\nu$. The space-translation on $\mathcal{A}$ is denoted by γ_x $(x \in \mathbf{Z}^\nu)$, which satisfies $\gamma_x(\mathcal{A}_\Lambda) = \mathcal{A}_{\Lambda+x}$ and the asymptotic abelianness property. A translation-invariant interaction Φ is a function from the finite $X \subset \mathbf{Z}^\nu$ into the selfadjoint elements of $\mathcal{A}$ such that $\Phi(X) \in \mathcal{A}_X$ and $\gamma_x(\Phi(X)) = \Phi(X + x)$ for all $X \subset \mathbf{Z}^\nu$ and $x \in \mathbf{Z}^\nu$. Then for each finite region Λ the local Hamiltonian $H(\Lambda)$ and the surface energy $W(\Lambda)$ are given as

$$H(\Lambda) = \sum_{X\subset\Lambda} \Phi(X), \qquad W(\Lambda) = \sum_{\substack{X\cap\Lambda\neq\emptyset \\ X\cap\Lambda^c\neq\emptyset}} \Phi(X).$$

We assume in the following that an interaction Φ is of relatively short range and of finite body; namely $\sum_{\Lambda\ni 0} ||\Phi(\Lambda)||/|\Lambda| < +\infty$ and there exists N_0 such that $\Phi(\Lambda) = 0$ if $|\Lambda| > N_0$. This is the case if Φ is of finite range, i.e. there exists d_0 such that $\Phi(\Lambda) = 0$ if $d(\Lambda) > d_0$ ($d(\Lambda)$ denotes the diameter of Λ).

Given an interaction Φ and an inverse temperature β (> 0) the local Gibbs state φ_Λ^c on a local algebra $\mathcal{A}_\Lambda$ is the canonical state:

$$\varphi_\Lambda^c(a) = \frac{\mathrm{Tr}_\Lambda a e^{-\beta H(\Lambda)}}{\mathrm{Tr}_\Lambda e^{-\beta H(\Lambda)}}, \qquad a \in \mathcal{A}_\Lambda,$$

where Tr_Λ is the canonical trace on $\mathcal{A}_\Lambda$. In order to state the Gibbs condition we recall the inner perturbation of a state. For a state φ on $\mathcal{A}$ and $h = h^* \in \mathcal{A}$ the perturbed state $[\varphi^h]$ is defined as the unique minimizer of the weakly* lower semicontinuous strictly convex functional $\omega \mapsto S(\omega, \varphi) + \omega(h)$ on the state space of $\mathcal{A}$, where $S(\omega, \varphi)$ is the relative entropy. Then φ is said to satisfy the Gibbs condition (with respect to Φ and β) if $[\varphi^{-\beta W(\Lambda)}]|\mathcal{A}_\Lambda = \varphi_\Lambda^c$ for any finite $\Lambda \subset \mathbf{Z}^\nu$. This definition of the Gibbs condition is a bit weaker than the usual one, but both are equivalent under the above assumption of Φ.

As thermodynamic limit $\Lambda \to \infty$ we consider the limit along the parallelepipeds; namely $\Lambda = \{x \in \mathbf{Z}^\nu : 0 \le x_i < a_i,\ i = 1, \ldots, \nu\}$ tends to infinity with $a_i \to \infty$ for $i = 1, \ldots, \nu$. (Indeed, most results below hold true in a more general limit of van Hove.)

In the above situation we state the central result concerning Gibbs states as follows. For a translation-invariant state φ of $\mathcal{A}$ the following three conditions are equivalent:

(i) φ satisfies the Gibbs condition with respect to Φ and β;

(ii) φ satisfies the KMS condition with respect to σ_t and β where

$$\sigma_t(a) = \lim_{\Lambda} e^{itH(\Lambda)} a e^{-itH(\Lambda)}, \qquad a \in \mathcal{A};$$

(iii) φ satisfies the variational principle:

$$s(\varphi) = \beta\varphi(A_\Phi) + p(\beta, \Phi),$$

where

$$s(\varphi) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} S(\varphi_\Lambda) \quad (\varphi_\Lambda = \varphi|\mathcal{A}_\Lambda),$$

$$A_\Phi = \sum_{\Lambda \ni 0} \frac{\Phi(\Lambda)}{|\Lambda|},$$

$$p(\beta, \Phi) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log \mathrm{Tr}_\Lambda e^{-\beta H(\Lambda)}.$$

($s(\varphi)$ is called the mean entropy and $p(\beta, \Phi)$ the pressure or the free energy.)

Via the GNS representation π_φ of $\mathcal{A}$ induced by the Gibbs state φ, we have a von Neumann algebra $\mathcal{M} = \pi_\varphi(\mathcal{A})''$ and a faithful normal state $\bar{\varphi}$ so that $\varphi = \bar{\varphi} \circ \pi_\varphi$. A net $\{x_j\}$ in $\mathcal{M}$ is said to converge almost uniformly to $x \in \mathcal{M}$ if for every $\varepsilon > 0$ there exists a projection $q \in \mathcal{M}$ such that $\bar{\varphi}(q) \ge 1 - \varepsilon$ and $||(x_j - x)q|| \to 0$. The following result resembles the McMillan theorem form information theory.

Theorem 1.1. *Assume that φ is an ergodic Gibbs state for Φ.*

(1)

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \pi_\varphi(-\log D_\Lambda) = s(\varphi) I \ \textit{strongly}.$$

(2) If $\nu = 1$ and Φ is of finite range, then

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \pi_\varphi(-\log D_\Lambda) = s(\varphi) I \ \textit{almost uniformly}.$$

(Indeed the almost uniform convergence holds true also when $\nu > 1$ but $\beta < \beta_0$ for some $\beta_0 > 0$ determined by Φ.)

The above (2) was proved in [6] by using the noncommutative ergodic theorem [10] and Araki's deep analysis on Gibbs states [2, 3].

We define for $0 < \varepsilon < 1$

$$\beta_\varepsilon(\varphi_\Lambda) = \min\{\log \mathrm{Tr}_\Lambda q : q \in \mathcal{A}_\Lambda \text{ is a projection with } \varphi(q) \geq 1 - \varepsilon\}.$$

If $\lambda_1 \geq \lambda_2 \geq \dots$ is the eigenvalue list of the density matrix of φ_Λ, then $\beta_\varepsilon(\varphi_\Lambda)$ is given by

$$\beta_\varepsilon(\varphi_\Lambda) = \log\left(\min\left\{N : \sum_{i=1}^{N} \lambda_i \geq 1 - \varepsilon\right\}\right).$$

The following illustrates the macroscopic uniformity which is a basic feature of statistical mechanical systems, and it resebles the asymptotic equipartition property of information theory.

Theorem 1.2. *Assume that φ is an ergodic Gibbs state for Φ.*

(1) For every $0 < \varepsilon < 1$

$$\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \beta_\varepsilon(\varphi_\Lambda) = s(\varphi).$$

(2) If the surface energies $W(\Lambda)$ are uniformly bounded, then for every $0 < \varepsilon < 1$

$$\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \beta_\varepsilon(\varphi_\Lambda) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \beta_\varepsilon(\varphi_\Lambda^c) = s(\varphi).$$

(Note in this case that the Gibbs state is unique, i.e. the phase transition does not occur.)

(3) If $\nu = 1$ and Φ is of finite range, then

$$\lim_{n \to \infty} \frac{\beta_\varepsilon(\varphi_{[1,n]})}{n} = \lim_{n \to \infty} \frac{\beta_\varepsilon(\varphi_{[1,n]}^c)}{n} = \lim_{n \to \infty} \frac{S(\varphi_{[1,n]})}{n} = \lim_{n \to \infty} \frac{S(\varphi_{[1,n]}^c)}{n}.$$

For a translation-invariant state ω, the limit

$$h(\omega|\beta, \Phi) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} S(\omega_\Lambda, \varphi_\Lambda^c)$$

exists due to the existence of the mean entropy and the pressure. In fact, we have

$$h(\omega|\beta, \Phi) = -s(\omega) + \beta\omega(A_\Phi) + p(\beta, \Phi),$$

and ω is a Gibbs state for Φ if and only of $h(\omega|\beta,\Phi)=0$.

For states ψ_1,ψ_2 on a matrix algebra $\mathcal{B}$, a variant of the relative entropy is defined as

$$\begin{aligned} S_{\mathrm{co}}(\psi_1,\psi_2) &= \max\Big\{\sum_i \psi_1(q_i)\log\frac{\psi_1(q_i)}{\psi_2(q_i)} : q_i \text{ are projections in } \mathcal{B},\ \sum_i q_i = I\Big\} \\ &= \max\{S(\psi_1|\mathcal{C},\psi_2|\mathcal{C}) : \mathcal{C} \text{ is a commutative } C^*\text{-subalgebra of } \mathcal{B}\}. \end{aligned}$$

The monotonicity of relative entropy implies $S_{\mathrm{co}}(\psi_1,\psi_2)\le S(\psi_1,\psi_2)$ and this inequality is strict except for the case of ψ_1,ψ_2 having the commuting densities. S_{co} may be related to measurements described by projection-valued measures.

Theorem 1.3. *Let ω be a translation-invariant state on $\mathcal{A}$.*

(1)

$$\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}S_{\mathrm{co}}(\omega_\Lambda,\varphi_\Lambda^{\mathrm{c}}) = h(\omega|\beta,\Phi).$$

(2) If $\nu=1$, Φ is of finite range, and φ is the Gibbs state for Φ, then

$$\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}S_{\mathrm{co}}(\omega_\Lambda,\varphi_\Lambda^{\mathrm{c}}) = \lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}S_{\mathrm{co}}(\omega_\Lambda,\varphi_\Lambda) = \lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}S(\omega_\Lambda,\varphi_\Lambda) = h(\omega|\beta,\Phi).$$

(Indeed this is true also when $\nu>1$ but $\beta<\beta_0$ for some $\beta_0>0$.)

In the case (2), the mean relative entropy

$$S_{\mathrm{M}}(\omega,\varphi) = \lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}S(\omega_\Lambda,\varphi_\Lambda)$$

exists, and the relative entropy S and its variant S_{co} have the same asymptotic limit. The particular case when φ is a product state was shown in [7].

The results in Section 1 were given in [8] except Theorem 1.1(2) in [6].

2. Entropy densities for algebraic states

An algebraic state is a translation-invariant state on one-dimensional quantum spin system $\mathcal{A}=\bigotimes_{x\in\mathbf{Z}}\mathcal{A}_x$, $\mathcal{A}_x=M_d(\mathbf{C})$, which is a slight generalization of "quantum Markov states" [1] and identical to "C^*-finitely correlated states" introduced in [5]. The definition is as follows: Let $\mathcal{B}$ be a finite-dimensional C^*-algebra, and let a completely positive unital map $\mathcal{E}: M_d(\mathbf{C})\otimes\mathcal{B}\to\mathcal{B}$ and a state ρ on $\mathcal{B}$ be given so that

$$\rho(\mathcal{E}(I\otimes b)) = \rho(b), \qquad b\in\mathcal{B}.$$

For any $a \in M_d(\mathbf{C})$ define $\mathcal{E}_a : \mathcal{B} \to \mathcal{B}$ by $\mathcal{E}_a(b) = \mathcal{E}(a \otimes b)$, $b \in \mathcal{B}$. Then the algebraic state φ generated by $(\mathcal{B}, \mathcal{E}, \rho)$ is given as

$$\varphi(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \rho(\mathcal{E}_{a_0} \circ \mathcal{E}_{a_1} \circ \cdots \circ \mathcal{E}_{a_n}(I))$$

for $a_0, \ldots, a_n \in M_d(\mathbf{C})$. Here the complete positivity of $\mathcal{E}$ ensures the positivity of φ.

The ergodicity and the strong mixing for algebraic states are characterized in several ways as follows. (1) was shown in [5].

Proposition 2.1. *Let φ be an algebraic state generated by $(\mathcal{B}, \mathcal{E}, \rho)$.*

(1) The following conditions are equivalent:

(i) φ is ergodic;

(ii) $\mathcal{E}_I$ is irreducible, i.e. I is the only eigenvector of $\mathcal{E}_I$ with respect to the eigenvalue 1.

(2) The following conditions are equivalent:

(i) φ is strongly mixing;

(ii) φ is weakly mixing;

(iii) φ is completely ergodic;

(iv) $\mathcal{E}_I$ is primitive, i.e. $\mathcal{E}_I^n$ is irreducible for all $n \in \mathbf{N}$;

(v) $\lim_{n\to\infty} \mathcal{E}_I^n(x) = \rho(x)I$ for all $x \in \mathcal{B}$.

Let ψ_1, ψ_2 be states on a matrix algebra $\mathcal{B}$. For $0 < \varepsilon < 1$ we define another type of relative entropy quantity as follows:

$$\beta_\varepsilon(\psi_1, \psi_2) = \inf\{\log \psi_2(q) : q \text{ is a projection in } \mathcal{B},\ \psi_1(q) \geq 1 - \varepsilon\}.$$

This quantity has a natural meaning from the viewpoint of quantum hypothesis testing (see [7]).

The following are our main results in [9].

Theorem 2.2. *Assume that φ be a strongly mixing algebraic state on $\mathcal{A}$. Let $\varphi_n = \varphi|\mathcal{A}_{[1,n]}$.*

(1) For every $0 < \varepsilon < 1$

$$\lim_{n\to\infty} \frac{1}{n} \beta_\varepsilon(\varphi_n) = s(\varphi).$$

(2) For every translation-invariant state ω on $\mathcal{A}$, the mean relative entropy $S_{\mathrm{M}}(\omega, \varphi)$ exists and

$$\lim_{n\to\infty} \frac{1}{n} S_{\mathrm{co}}(\omega_n, \varphi_n) = S_{\mathrm{M}}(\omega, \varphi).$$

(3) For every completely ergodic state ω on $\mathcal{A}$ and $0 < \varepsilon < 1$

$$\limsup_{n\to\infty} \frac{1}{n}\beta_\varepsilon(\omega_n, \varphi_n) \le -S_{\mathrm{M}}(\omega, \varphi),$$

$$\liminf_{n\to\infty} \frac{1}{n}\beta_\varepsilon(\omega_n, \varphi_n) \ge -\frac{1}{1-\varepsilon}S_{\mathrm{M}}(\omega, \varphi).$$

The above (3) shows that we obtain $\exp\{\frac{1}{n}\beta_\varepsilon(\omega_n, \varphi_n)\} \approx \exp\{-S_{\mathrm{M}}(\omega, \varphi)\}$ for large n and small ε. To prove the theorem, we need the next result which says that strongly mixing algebraic states have a certain property of approximately product type.

Proposition 2.3. *Let φ be the algebraic state generated by $(\mathcal{B}, \mathcal{E}, \rho)$. Then φ is strongly mixing if and only if for any $\alpha > 1$ there exists $l \in \mathbf{N}$ such that for all $m \in \mathbf{N}$*

$$\begin{aligned}\alpha^{-1}(\varphi|\mathcal{A}_{(-\infty,m]}) \otimes (\varphi|\mathcal{A}_{[m+l+1,\infty)}) &\le \varphi|\mathcal{A}_{(-\infty,m]\cup[m+l+1,\infty)} \\ &\le \alpha(\varphi|\mathcal{A}_{(-\infty,m]}) \otimes (\varphi|\mathcal{A}_{[m+l+1,\infty)}).\end{aligned}$$

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